A Very Strong Higher Guessing Model Principle

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- *Martin's Axiom*: c.c.c. forcings and κ arbitrary.
- Increasing κ or enlarging $\mathscr{C}$ yields stronger axioms.

$$\mathrm{MM} \Leftrightarrow \mathrm{SPFA} \Rightarrow \mathrm{PFA} \Rightarrow \mathrm{MA}_{\omega_1}$$

Old but gold

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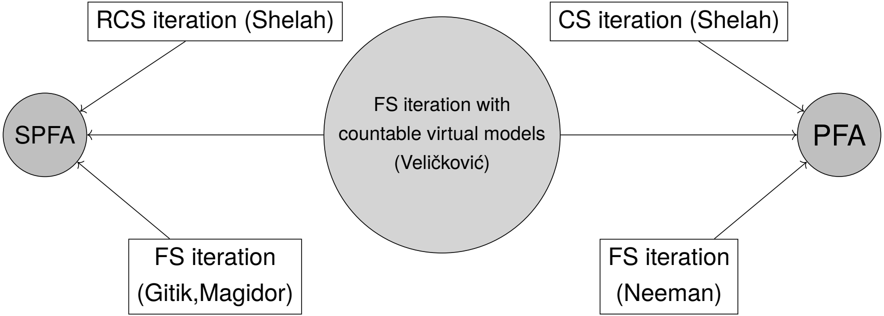
Do we have forcing axioms implying certain statements like $|\mathbb{R}| \geq \omega_3$, $\mathrm{TP}(\omega_3)$, SCH, etc.?

Iteration and forcing axioms



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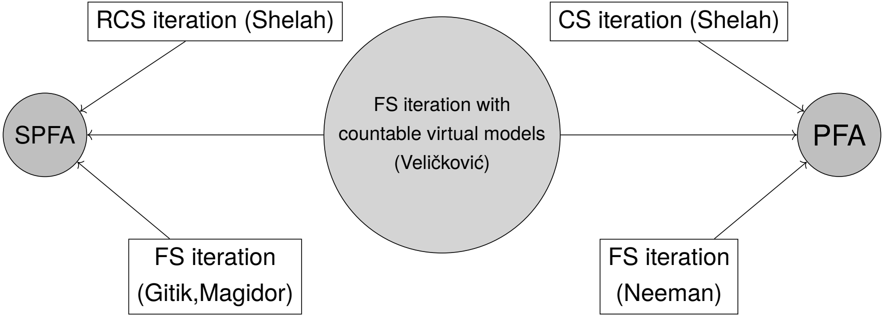

No iteration theorem for stationary-set preserving forcings.

Every maths talk needs a joke

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Basically, it's the axiom we can trust to have our back.

I will discuss certain consistency results that could be regarded as potential consequences of an imaginary higher forcing axiom.

Guessing models

Definition (Viale, Weiss)

$M \prec \mathcal{H}(\theta)$ is a κ-*guessing model* if for every $x \in \bigcup_{X \in M} \mathcal{P}(X)$ the following are equivalent.

1. $\exists x^* \in M$ so that $x \cap M = x^* \cap M$,
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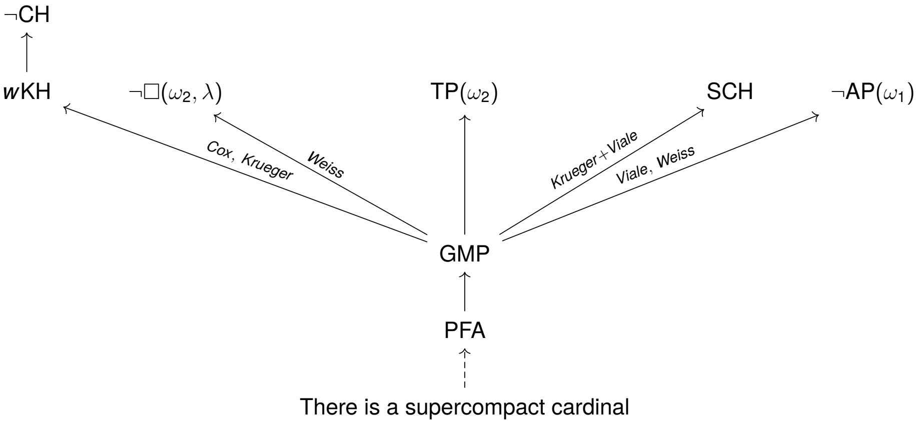
Theorem (Viale, Weiss)

PFA implies GMP: $\{M \prec \mathcal{H}(\theta) : |M| = \omega_1 \wedge M$ *is* ω_1*-guessing* $\}$ *is stationary for every* $\theta \geq \omega_2$*.*

Consequences



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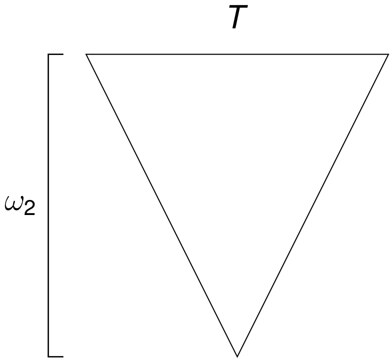

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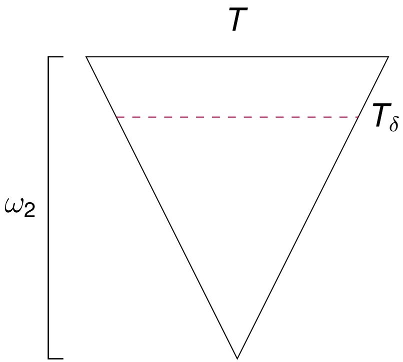

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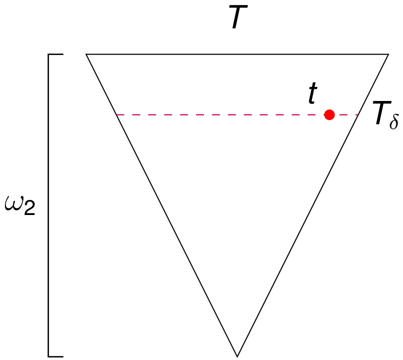

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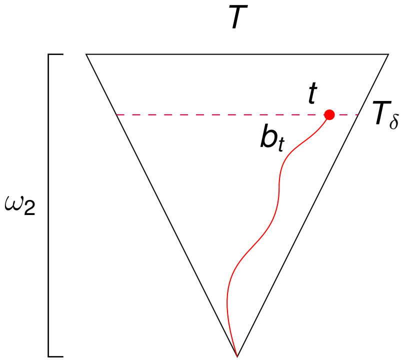

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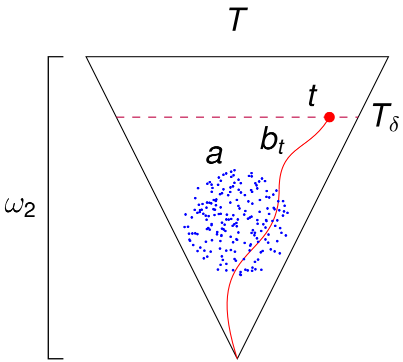

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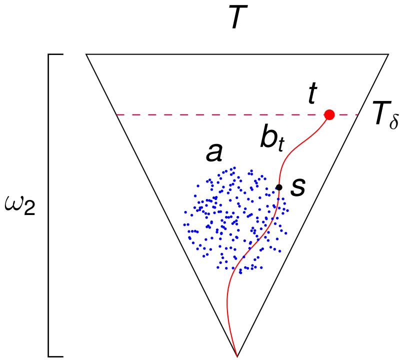

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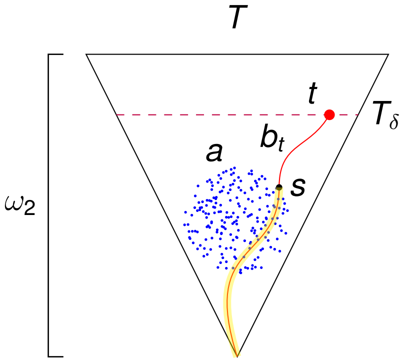

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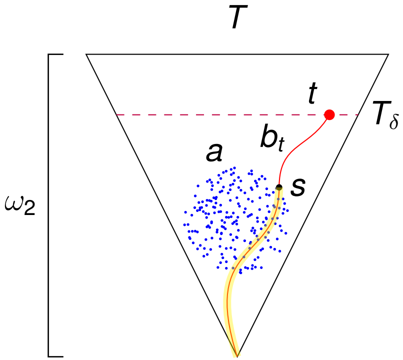

$a \cap b_t = a \cap b_s \in M$

Approachability

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$\delta < \omega_2$ is *approachable by $\vec{a}$* if there is $c \subseteq \delta$ such that

1. $\mathrm{ot}(c) < \delta$.
2. $\forall \xi < \delta \, \exists \zeta < \delta \; a_\xi \cap c = a_\zeta$.

Approachability ideal

The *approachability ideal* $I[\omega_2]$ is generated by $\mathfrak{J}_{\rm ns}(\omega_2)$ and sets of the form

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$\mathrm{AP}(\omega_1)$ states that $\omega_2 \in I[\omega_2]$.

GMP $\Rightarrow \neg$AP(ω_1)

Proof. There are stationarily many non-approachable points of cofinality ω_1.
Fix an approaching sequence $\vec{b} = \langle b_\xi : \xi < \omega_2 \rangle$.

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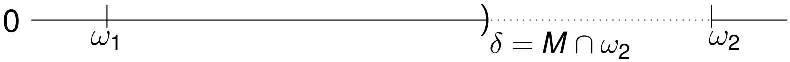

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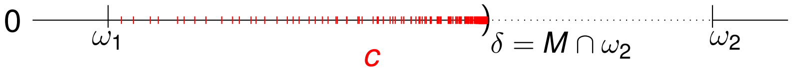

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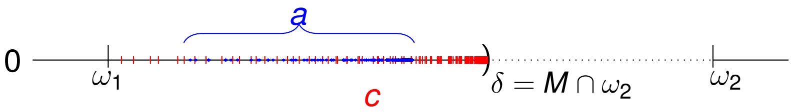

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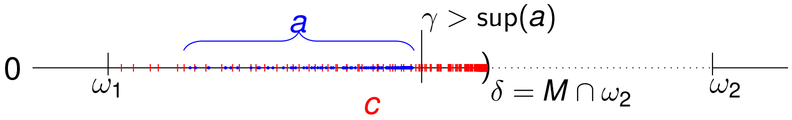

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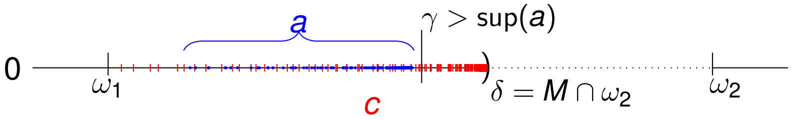

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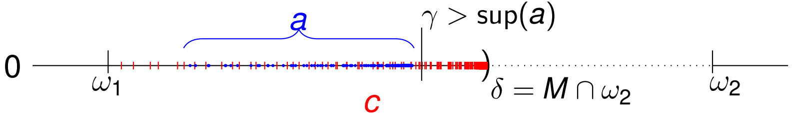

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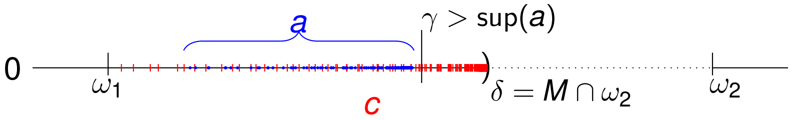

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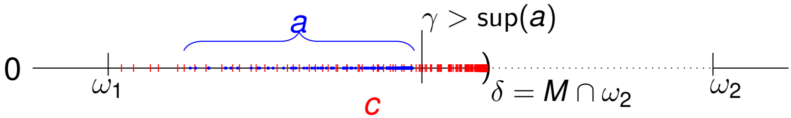

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Let T $:=$ ZFC together with the statement that for every $\theta \geq \omega_3$, the set

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is stationary in $\mathcal{P}_{\omega_3}(\mathcal{H}(\theta))$.

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Theorem (Trang)

1. $\mathrm{Con}(\mathit{ZFC} + \exists$ *s.c. cardinal*$) \Rightarrow \mathrm{Con}(T)$.
2. *Assume T. There is a transitive model* $M \models$ *"*$AD_{\mathbb{R}} + \Theta$ *is regular" with* $\mathbb{R} \subseteq M$.

GMP$^+$

Definition

$M \prec \mathcal{H}(\theta)$ of size ω_2 is a *strongly* ω_1-guessing model if it is the union of an ω_1-closed $\in$-sequence of ω_1-guessing models of size ω_1.

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Definition (GMP^+)

For every $\theta \geq \omega_3$, there are stationarily many strongly ω_1-guessing elementary submodels of $\mathcal{H}(\theta)$

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M., Veličković

GMP$^+$ implies $I[\omega_2] = \mathfrak{I}_{\mathrm{ns}}(\omega_2)$ mod $\mathrm{Cof}(\omega_1)$.

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Theorem (Mitchell)

$I[\omega_2] = \mathfrak{J}_{ns}(\omega_2)$ mod $\mathrm{Cof}(\omega_1)$ is consistent.

GMP^+

M., Veličković

$\mathrm{Con}(\mathrm{ZFC} + \text{ there are two s.c. cardinals}) \Rightarrow \mathrm{Con}(\mathrm{GMP}^+)$.

The proof uses forcing with pure side conditions.

GMP^+

M., Veličković

$\mathrm{Con}(\mathsf{ZFC} +$ there are two s.c. cardinals$) \Rightarrow \mathrm{Con}(\mathsf{GMP}^+)$.

The proof uses forcing with pure side conditions. Let $\kappa < \lambda$ be supercompact cardinals. The conditions are finite sets of countable and κ-Magidor virtual models which satisfy certain requirements with respect to $\{\alpha < \lambda : V_\alpha \prec V_\lambda\}$.

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Note that $I[\omega_2] = \mathfrak{J}_{\mathrm{ns}}(\omega_2) \bmod \mathrm{Cof}(\omega_1)$ implies $|\mathbb{R}| \geq \omega_3$ (due to Shelah.)

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Note that $I[\omega_2] = \mathfrak{J}_{\mathrm{ns}}(\omega_2) \bmod \mathrm{Cof}(\omega_1)$ implies $|\mathbb{R}| \geq \omega_3$ (due to Shelah.) So

- GMP$^+ \Rightarrow |\mathbb{R}| \geq \omega_3$.

Generalization

GMP(κ, γ) states that for every sufficiently large regular cardinal θ, the set

$$\mathcal{G}_{\kappa,\gamma}(\mathcal{H}(\theta)) := \{M \prec \mathcal{H}(\theta) : |M| < \kappa \wedge M \text{ is } \gamma\text{-guessing}\}$$

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is stationary in $\mathcal{P}_\kappa(\mathcal{H}(\theta))$.
GMP$^+(\kappa, \gamma)$ is defined in the obvious way. So GMP $\equiv$ GMP(ω_2, ω_1) and GMP$^+$ $\equiv$ GMP$^+(\omega_3, \omega_1)$.

- The appropriate generalizations of the consequences of GMP or GMP$^+$ follow from their corresponding higher principles.

IGMP

Definition (Cox, Krueger)

A γ-guessing model is called *indestructible* if it remains guessing in any outer transitive universe in which γ is a cardinal.

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Theorem (Cox, Krueger)

PFA $\Rightarrow$ *IGMP.*

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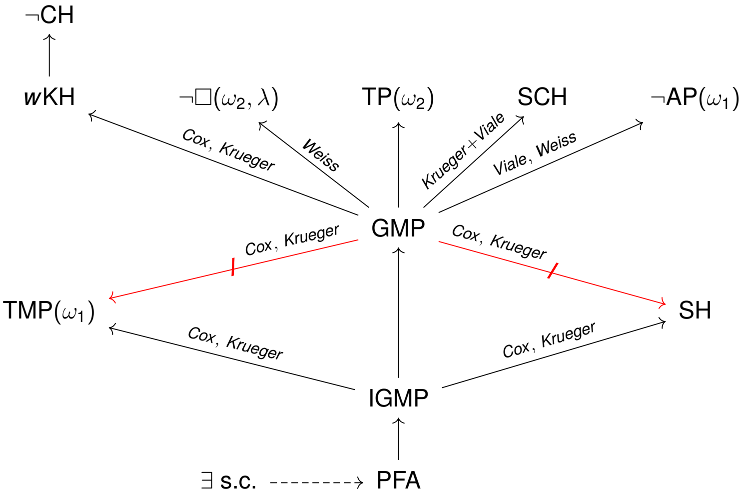
Idea: Iterate standard specializing forcing up to a supercompact cardinal κ using *finite conditions* and then add an arbitrarily large number of reals. The iteration is technical and delicate. The main difficulty is performing a quotient analysis for models of size less than κ (the so-called κ-Magidor models).


¬CH
wKH
¬□(ω2, λ)
TP(ω2)
SCH
¬AP(ω1)
Cox, Krueger
Weiss
Krueger+Viale
Viale, Weiss
GMP
Cox, Krueger
Cox, Krueger
TMP(ω1)
SH
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IGMP
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TMP

Todorčević Maximality Principle.

Definition (TMP(κ^+))

Every forcing that adds a new subset of κ^+ whose initial segments are in the ground model must collapse some cardinal.

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Theorem (Todorčević, $2^{\aleph_0} < \aleph_{\omega_1}$)

Suppose that every tree of size and height ω_1 without cofinal branches is special. Then TMP(ω_1) holds.

Theorem (Cox, Krueger, and $2^{\aleph_0} < \aleph_{\omega_1}$)

IGMP $\Rightarrow$ *TMP*(ω_1)

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Theorem (Golshani, Shelah)

TMP(κ^+) *is forceable, assuming suitable large cardinals.*

IGMP$^+$

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Definition

M $\prec \mathcal{H}(\theta)$ of size ω_2 is an *indestructible strongly* ω_1*-guessing model* if it is the union of an ω_1-closed $\in$-sequence of indestructible ω_1-guessing models of size ω_1.

IGMP^+

M., Veličković

Suppose that $V \subseteq W$ are transitive inner models of ZFC.

- $V \models$ "$\mathrm{IGMP}^+ + 2^{\omega_1} < \aleph_{\omega_2}$".
- $\mathcal{P}(\omega_2)^V \neq \mathcal{P}(\omega_2)^W$.

Then either $\mathcal{P}(\omega_1)^V \neq \mathcal{P}(\omega_1)^W$ or some V-cardinal $\leq 2^{\omega_1}$ is no longer a cardinal in W.

IGMP$^+$

Corollary (IGMP$^+$ $+ 2^{\omega_0} < \aleph_{\omega_1} + 2^{\omega_1} < \aleph_{\omega_2}$)

Let $W \models$ "ZFC" be a transitive extension with the same cardinals and reals. Then $\mathcal{P}(\omega_2)^V = \mathcal{P}(\omega_2)^W$.

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Corollary ($2^{\aleph_0} < \aleph_{\omega_1} + 2^{\aleph_1} < \aleph_{\omega_2}$)

IGMP^+ implies both $\mathrm{TMP}(\omega_1)$ and $\mathrm{TMP}(\omega_2)$.

$\mathrm{Con}(\mathsf{IGMP}^+)$

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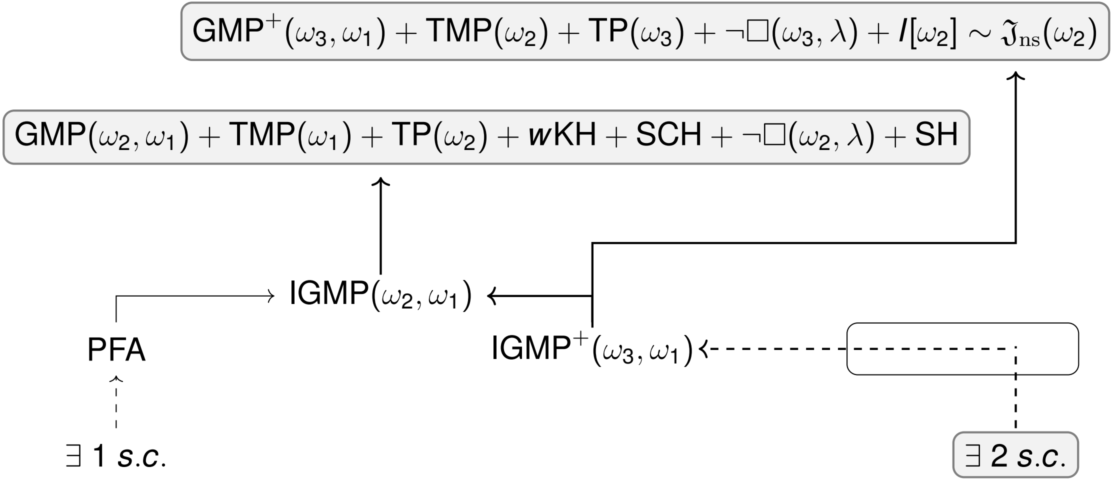
$\mathrm{GMP}^+(\omega_3,\omega_1) + \mathrm{TMP}(\omega_2) + \mathrm{TP}(\omega_3) + \neg\Box(\omega_3,\lambda) + I[\omega_2] \sim \mathfrak{J}_{\mathrm{ns}}(\omega_2)$
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$\mathrm{IGMP}(\omega_2,\omega_1)$
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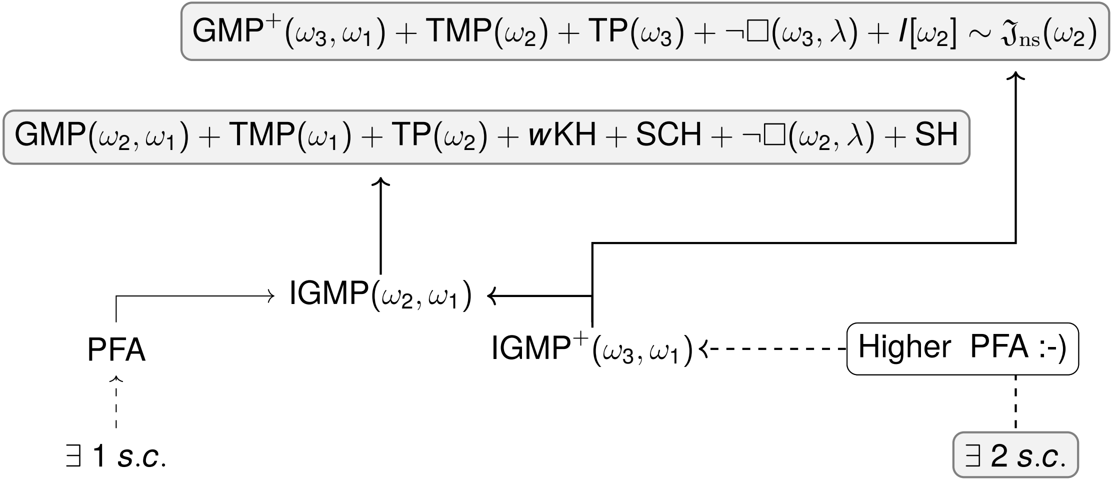
$\exists$ 1 *s.c.*
$\exists$ 2 *s.c.*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$\mathrm{IGMP}(\omega_2, \omega_1)$
PFA
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Higher PFA :-)
$\exists$ 1 *s.c.*
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