Finite Support Product of Strongly Proper Forcings

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Abstract

Given a sequence $\vec{\mathbb{P}}$ of strongly proper forcings, we use models as side conditions to construct a scaffolding forcing $\mathbb{M}(\vec{\mathbb{P}})$ so that forcing with it adds V-generic filters for finite sub-products of $\vec{\mathbb{P}}$.

Let X be a set and let $\mathbb{P}$ be a forcing. A condition $p \in \mathbb{P}$ is said to be strongly $(X, \mathbb{P})$-generic if for every $q \leq p$, there is a condition $q \restriction X \in \mathbb{P} \cap X$ such that every $r \in X \cap \mathbb{P}$ is compatible with q. For a collection $\mathcal{S}$ of sets, we say $\mathbb{P}$ is $\mathcal{S}$-strongly proper, if for every $X \in \mathcal{S}$ and every $p \in \mathbb{P} \cap X$, there is a strongly $(X, \mathbb{P})$-generic condition $q \leq p$. These definitions are due to Mitchell [1]. We call $q \restriction X$ a projection of q to X. It is easily seen that the product of finitely many $\mathcal{S}$-strongly proper forcings is $\mathcal{S}$-strongly proper. In fact, if $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ are strongly $(X, \mathbb{P})$-generic and $(X, \mathbb{Q})$-generic, respectively, then (p, q) is strongly $(X, \mathbb{P} \times \mathbb{Q})$-generic. However, it is easy to see that finite support products of $\mathcal{S}$-strongly proper forcings are not in general $\mathcal{S}$-strongly proper forcing.

Definition 1. *Let $\vec{\mathbb{P}} = \langle \mathbb{P}_i : i \in I \rangle$ be a sequence of forcings and let also θ be a sufficiently large regular cardinal with $\vec{\mathbb{P}} \in H_\theta$. Assume that $\mathcal{S}$ is a collection of elementary submodels M of H_θ with $\vec{\mathbb{P}} \in M$ such that for every $N, P \in \mathcal{S}$, if $N \in P$, then $N \subseteq P$. We let $\mathbb{M} := \mathbb{M}(\mathcal{S}, \vec{\mathbb{P}})$ consist of conditions $p = (\mathcal{M}_p, w_p)$ such that*

1. *$\mathcal{M}_p \subseteq \mathcal{S}$ is a finite $\in$-chain, and*
2. *$w_p : \mathrm{dom}(w_p) \to H_\theta$ is a finite function such that for every $i \in \mathrm{dom}(w_p)$, $w_p(i) \in \mathbb{P}_i$ and, moreover, for every $M \in \mathcal{M}_p$ with $i \in M$, $w_p(i)$ is strongly $(M, \mathbb{P}_i)$-generic.*

We say q is stronger than p and write $q \leq p$ if and only if $\mathcal{M}_q \supseteq \mathcal{M}_p$, and $w_q(i) \leq_{\mathbb{P}_i} w_p(i)$, for every $i \in \mathrm{dom}(w_p)$.

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Lemma 2. *Let $M \in \mathcal{S}$. Assume that $p \in M \cap \mathbb{M}$. Then there is a condition $q \leq p$ with $M \in \mathcal{M}_q$.*

Proof. Let q be defined as follows. Set $\mathcal{M}_q := \mathcal{M}_p \cup \{M\}$ which is a finite $\in$-chain. Note that $\mathrm{dom}(w_p) \subseteq M$. For each $i \in \mathrm{dom}(w_p)$, we can extend $w_p(i)$ to a strongly $(M, \mathbb{P}_i)$-generic condition $z_i \in \mathbb{P}_i$. Now let w_q be defined on $\mathrm{dom}(w_p)$ by letting $w_q(i) = z_i$. Notice that z_i is also $(N, \mathbb{P}_i)$-strongly generic for every model N in $\mathcal{M}_p$ with $i \in N$. Thus q is a condition. It is clear that $q \leq p$. $\boxed{2}$

Lemma 3. *Suppose that p is a condition in $\mathbb{M}$. Let $M \in \mathcal{M}_p$. Then p is strongly $(M, \mathbb{M})$-generic.*

Proof. Define $p \restriction M := (\mathcal{M}_{p\restriction M}, w_{p\restriction M})$ by letting $\mathcal{M}_{p\restriction M} := \mathcal{M}_p \cap M$ and defining $w_{p\restriction M}$ on $\mathrm{dom}(w_p) \cap M$ by letting $w_{p\restriction M}(i)$ be some projection of $w_p(i)$ to M, say $w_p(i) \restriction M$, which exists by the fact that such $w_p(i)$ is strongly $(M, \mathbb{P}_i)$-generic. Notice that $p \restriction M$ belongs to M. Suppose that q is a condition in $\mathbb{P} \cap M$ extending $p \restriction M$. Define a common extension, say r, of p and q as follows. Let $\mathcal{M}_r$ be $\mathcal{M}_p \cup \mathcal{M}_q$ which is easily seen to be a finite $\in$-chain. Define w_r on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$ by

$$w_r(i) = \begin{cases} w_p(i) & i \notin M, \\ z_i & i \in \mathrm{dom}(w_p) \cap M, \\ u_i & i \in \mathrm{dom}(w_q) \setminus \mathrm{dom}(w_p), \end{cases}$$

where z_i is a common extension of $w_q(i)$ and $w_p(i)$ (such condition exists as $w_q(i) \leq w_p(i) \restriction M$), and $u_i \leq w_q(i)$ is a condition which is generic for every $N \in \mathcal{M}_r$ with $N \supseteq M$. Note that such a condition exists, as models in $\mathcal{M}_p \setminus \mathcal{M}_q$ forms a $\subseteq$-sequence, by our assumption on $\mathcal{S}$ and $\mathcal{M}_p$. Thus one can inductively extend $w_q(i)$ to find a condition $u_i \leq w_q(i)$ which is $(N, \mathbb{P}_i)$ or every model in $\mathcal{M}_p \setminus \mathcal{M}_q$, and hence is $(N, \mathbb{P}_i)$-generic for all relevant models. It is easy to see that r is a condition in $\mathbb{M}$ with $r \leq p, q$. $\boxed{3}$

Proposition 4. $\mathbb{M}$ *is $\mathcal{S}$-strongly proper.*

Proof. By Lemmas 2 and 3. $\boxed{4}$

Lemma 5. *Assume $a \subseteq I$ is a finite. Then the set of conditions $p \in \mathbb{M}$ with $a \subseteq \mathrm{dom}(p)$ is dense in $\mathbb{M}$.*

Proof. Suppose p is a condition. By Lemma 2, we may assume that there is some model in $\mathcal{M}_p$ which contains a. For every $i \in a \setminus \mathrm{dom}(w_p)$, let $M^i \in \mathcal{M}_p$ be the least model with $i \in M^i$. Let q be defined by letting $\mathcal{M}_q := \mathcal{M}_p$ and extending w_p to w_q on $\mathrm{dom}(w_p) \cup a$ so that for every $i \in a \setminus \mathrm{dom}(w_p)$, $w_q(i)$ is strongly $(M, \mathbb{P}_i)$-generic, for every $M \in \mathcal{M}_p$ with $i \in M$. This is possible as i belongs to each model above M^i, thus we can inductively extend $\mathbf{1}_{\mathbb{P}_i}$ to a condition $w_q(i)$ which is strongly $(M, \mathbb{P}_i)$-generic for every $M \in \mathcal{M}_p$ containing i. $\boxed{5}$

Proposition 6. *Let G be a V-generic filter on $\mathbb{M}$. Assume that a is a finite subset of I. Then $G_a := \{w_p \restriction a : p \in G\}$ is a V-generic filter over $\prod_{i \in a} \mathbb{P}_i$.*

Proof. By Lemma 5, G_a is nonempty. It is clear that G_a is a filter. If $D \in V$ is a dense subset of $\prod_{i\in a} \mathbb{P}_i$, then $E := \{r \in \mathbb{M} : w_r \restriction a \in D\}$ is dense. To see this, fix q in $\mathbb{M}$. By Lemma 5, we may assume $a \subseteq \mathrm{dom}(w_q)$, then let $w \in D$ be such that $w \leq_{\prod_{i\in a} \mathbb{P}_i} w_q \restriction a$, we can now define r by letting $\mathcal{M}_r = \mathcal{M}_q$ and letting w_r be defined the same as w on a, and otherwise let it be the same as w_q. Clearly r is an extension of q which belongs to E. Thus E is a dense subset of $\mathbb{M}$. Therefore there is some $p \in G \cap E$, which in turn implies $w_p \restriction a \in G_a \cap D$. Thus G_a is V-generic. $\boxed{6}$

Remark 7. In fact, the mapping $p \mapsto w_p \restriction a$ is a projection from $\mathbb{M}$ to $\prod_{i\in a} \mathbb{P}_i$.

References

[1] William J. Mitchell. Adding closed unbounded subsets of ω_2 with finite forcing. *Notre Dame J. Formal Logic*, 46(3):357–371, 07 2005.