Příkrý Forcing and Properness

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Abstract

It is shown the Příkrý forcing for measurable κ is proper below V_κ.

Let $\mathcal{U}$ be a measure on κ. The Příkrý forcing $\mathbb{P}_\mathcal{U}$ consists of conditions $p := (A_p, s_p)$, where

1. $A_p \in \mathcal{U}$,
2. $s_p \in [\kappa]^{<\omega}$, and
3. $\max(s_p) < \min(A)$.

A condition p is stronger than q, i.e. $p \leq q$, if $A_p \subseteq A_q$, $s_p \cap \max(s_q) = s_q$, and $s_p \setminus s_q \subseteq A_q$.

Proposition 0.1. *For every regular $\lambda > 2^{\kappa^+}$, every $X \in V_\kappa$, every condition p, and every $M \prec H_\lambda$ with $p, X, \mathcal{U} \in M$ of size $< \kappa$, there is $q \leq p$ such that q is $(M, \mathbb{P}, X)$-generic, i.e.,*

$$q \Vdash_{\mathbb{P}_\mathcal{U}} \text{“}M[\dot{G}] \cap X = M \cap X\text{”}.$$

Proof. Let $p := (A_p, s_p)$. We first define a condition p' by letting

$$A_{p'} := \bigcap (M \cap \mathcal{U}),$$

and $s_{p'} := s_p$. By κ-completeness of $\mathcal{U}$, $A_{p'} \in \mathcal{U}$. Clearly $p' \leq p$. Let Ω be the set of pairs $(\dot{\tau}, x)$ such that

- $\dot{\tau}$ is a $\mathbb{P}_\mathcal{U}$-name in M,
- $x \in X$, and
- some condition below p forces $\dot{\tau} = \check{x}$.

Clearly $|\Omega| < \kappa$. Now by the Příkrý condition[1] and the κ-completeness of $\mathcal{U}$, there is $B \subseteq A_{p'}$ such that $q := (B, s_p)$ decides $\dot{\tau} = \check{x}$, for every $(\dot{\tau}, x) \in \Omega$. We claim that q is as required.

Suppose that $\dot{\sigma}$ is forced by some condition $q' \leq q$ to be in $M[\dot{G}] \cap X$. We may extend q' to some condition q'', find a $\mathbb{P}_\mathcal{U}$-name $\dot{\tau} \in M$, and an element $x \in X$ such that q'' forces $\dot{\tau} = \dot{\sigma} = \check{x}$. Therefore, $(\dot{\tau}, x) \in \Omega$, since $q'' \leq p$. On the other hand, q must decide $\dot{\tau} = \check{x}$ in the same way, that is $q \Vdash$ “$\dot{\tau} = \check{x}$”. Notice that q is a direct extension of p. So by elementarity, there are $A' \in \mathcal{U} \cap M$ and $x' \in X \cap M$ such that (A', s_p) forces $\dot{\tau} = \check{x}'$. But $q \leq (A', s_p)$, and hence $x' = x$. Therefore, $q'' \Vdash \dot{\sigma} \in M \cap X$. Since $\dot{\sigma}$ was arbitrary, we have

$$q \Vdash_{\mathbb{P}_\mathcal{U}} \text{“}M[\dot{G}] \cap X = M \cap X\text{”}.$$

□

Remark 0.2. *$\mathbb{P}_\mathcal{U}$ preserves stationary sets in V_κ.*

Remark 0.3. *Letting $X = \omega_1$ and M be countable, we have that $\mathbb{P}_\mathcal{U}$ is semi-proper.*

[1] For every statement ϕ in the forcing language of $\mathbb{P}_\mathcal{U}$ and for every $p \in \mathbb{P}_\mathcal{U}$, there is $q \leq p$ with $s_q = s_p$ that decides ϕ.