A Proof of the Δ-System Lemma Using Elementary Submodels

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The Δ-System Lemma

The well-known Δ-system Lemma states that if the size of a family $\mathcal{F}$ of sets is sufficiently large relative to the size of its members, then there should be a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $|\mathcal{F}|$ such that, letting $r = \bigcap \mathcal{F}'$, then every two distinct members of $\mathcal{F}'$ are disjoint modulo r.

Definition 1. *A family $\mathcal{F}$ of sets forms a Δ-system, if there is r such that for every $x \neq y$ in $\mathcal{F}$, $x \cap y = r$. Then, r is said to be the root of $\mathcal{F}$.*

Lemma 2 (Δ-system Lemma)**.** *Let θ be a regular uncountable cardinal and let $\kappa < \theta$ be a cardinal. Suppose that $\mathcal{F}$ is a set of size θ such that every $x \in \mathcal{F}$ is of size less than κ. Assume that for every cardinal $\lambda < \theta$, $\lambda^{<\kappa} < \theta$. Then there is a Δ-system $\mathcal{F}' \subseteq \mathcal{F}$ of size θ.*

Proof. Without loss of generality, we may assume that $\bigcup \mathcal{F} \subseteq \theta$, which implies that $\mathcal{F} \in H_{\theta^+}$.

Claim 3. *There is an $M \prec H_{\theta^+}$ of size less than θ and closed under $<\kappa$-sequences so that $\kappa, \theta, \mathcal{F} \in M$ and $M \cap \theta \in \theta$.*

Proof. By our cardinal arithmetic assumption, we can pick an infinite regular cardinal λ in the interval $[\kappa, \theta)$. Let us construct a sequence $\langle M_\alpha : \alpha \leq \lambda \rangle$ as follows. Let $M_0 \prec H_{\theta^+}$ be countable with $\kappa, \theta, \mathcal{F} \in M_0$. Assume that the sequence has been constructed up to $\alpha \leq \kappa^+$. If α is limit, let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. If $\alpha = \beta + 1$, then pick $M_\alpha \prec H_\theta$ of size less than θ such that

$$[M_\beta]^{<\kappa} \cup \sup(\mathrm{M}_\beta \cap \theta) \subseteq M_\alpha.$$

This is possible by the cardinal arithmetic assumption and the fact that θ is regular. Observe that M_λ is of size less than θ. We shall show that M_λ is as required. By the construction and the fact that λ is regular, M_λ is closed under $<\kappa$-sequences and $\kappa, \theta, \mathcal{F} \in M_\lambda$. To see that $M_\lambda \cap \theta \in \theta$,

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let $\xi \in M_\lambda \cap \theta$, then $\xi \in M_\alpha \cap \theta$, for some $\alpha < \lambda$. Thus $\xi \subseteq M_{\alpha+1}$, by the construction of $M_{\alpha+1}$. Therefore, $M_\lambda \cap \theta$ is an ordinal.

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Returning to the main proof, pick an elementary submodel $M \prec H_{\theta^+}$ as in the above claim. Since M is of size less than θ, there is $x \in \mathcal{F} \setminus M$. Fix such an x and set $r = x \cap M$. We have $r \in M$, by the closure property of M. Now, for each $\alpha \in M \cap \theta$ with $\alpha > \sup(r)$, there exists $x_\alpha \in \mathcal{F} \cap M$ such that

$(*)$ $x_\alpha \cap \alpha = r$, and

$(\dagger)$ $x_\alpha \setminus \alpha \neq \varnothing$.

To see this, observe that x satisfies the above properties, and hence by elementarity such a set exists in $M \cap \mathcal{F}$. By the Axiom of Choice in M, there is a sequence $s := \langle x_\alpha : \sup(r) < \alpha < \theta \rangle \in M$ so that $x_\alpha \in \mathcal{F}$ satisfies $(*)$ and $(\dagger)$. Now, as θ is regular, one can easily construct a subsequence $\langle x_{\alpha_\gamma} : \gamma < \theta \rangle$ of s such that if $\gamma < \eta$, then $x_{\alpha_\gamma} \subseteq \alpha_\eta$. Moreover, by elementarity such a subsequence exists in M. Suppose that $\gamma < \eta \in M \cap \theta$, we have

$$x_{\alpha_\gamma} \cap x_{\alpha_\eta} = x_{\alpha_\gamma} \cap x_{\alpha_\eta} \cap \alpha_\eta = x_{\alpha_\gamma} \cap r = r.$$

Thus by elementarity, $\mathcal{F}' = \{x_{\alpha_\gamma} : \gamma < \theta\}$ forms a Δ -system lemma with the root r. On the other hand, by $(\dagger)$, we have $|\mathcal{F}'| = \theta$.

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